CERTAIN INVARIANT SEQUENCES OF POLYNOMIALS*

BY

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We discuss certain sequences of uniform functions of one variable such that the derivative of each term of a given sequence is equal to the preceding term, and each term is changed by a linear transformation on the variable into a multiple of itself, the multiplier being a function of the rank of the term alone, and the linear transformation and the multiplier function being the same for all terms. It is an interesting problem to determine all such sequences of a certain type, described presently, and to assign the corresponding transformations and multipliers. This is done in the following sections; there is an infinity of solutions. Although we consider only functions of one variable, the method is general and applicable to functions of any number of variables.

The terms are necessarily polynomials. By linear transformations on the rank and the variable, any number of distinct sequences having the stated properties for any given linear transformations on their variables can be replaced by new sequences, all of which are transformed alike by the same linear transformation on the variable, and hence are instances of a single sequence of the original kind. In this respect the theory of any number of distinct sequences of the kind described can be unified.

The determination of *all* sequences of the general kind just mentioned, presents no difficulty. The solution, if no restriction be imposed on the numbers defined by the sequences, contains an infinity of arbitrary constants. From our point of view, which is to extend in as simple a manner as possible the existing instances of such sequences, the perfectly general solution is of but slight interest; it is difficult to see how it could lead to interesting extensions of the known cases. This remark is the origin of §4, Theorem 3, where the alternative condition $\psi_0\psi_2-\psi_1^2=0$ is rejected, and its contradictory leads to a definite infinity of solutions, each of which contains only a finite number of arbitrary constants. We have endeavored to construct the theory so that its interest shall be arithmetical rather than algebraic.

There are four classic instances of such sequences. These will be derived in §8 as immediate special cases, to provide checks on the general theorems. The main points of the paper are the definitions of index and characteristic in §1, and the Theorems 1–11.

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Both in the classic instances and in their extensions, the sequences appear as particular solutions of what is here called the functional equation of invariant generators. This functional equation alone is not sufficient to define a particular sequence. The manner in which particular solutions are completely specified is explained in §§8, 9. For example, the Bernoulli numbers being the coefficients in the successive polynomials of one classic sequence, we use the even suffix notation $B_0, B_2, B_4, B_6, \cdots$, in which, if no further condition be imposed, $B_1, B_3, B_5, \cdots$ are entirely arbitrary. To eliminate this undesirable infinity of arbitrary constants, we adjoin to the functional equation another for the numbers B_{2n} $(n=0, 1, \cdots)$, in a perfectly definite manner, which completely defines the sequence, and so in all cases.

1. **Invariant sequences.** Let n be an integer $\geqq 0$, and x a real or complex variable. A statement involving n shall signify the totality of statements obtained from the given one by taking $n=0, 1, 2, \cdots$, successively, so that it will be unnecessary in formulas and elsewhere to indicate the range of n.

The notation $(a_1, \cdots, a_r)=(b_1, \cdots, b_r)$ means $a_i=b_i (i=1, \cdots, r)$; while $(a_1, \cdots, a_r)\neq(b_1, \cdots, b_r)$ means that at least one of $a_i=b_i$ $(i=1, \cdots, r)$ is false.

Let $f_n(x)$ be a single-valued function of x defined for all x as above, and, when necessary, impose the convention that $f_{-1}(x)$ is defined and finite for all values of x considered. If $f_n(x)$ is a polynomial in x, its degree in x is by definition n. If y is a function of x, the derivative of $f_n(y)$ with respect to x will be denoted by $f_n'(y)$. It is necessary to assume that $f_n'(y)$ exists only in what immediately follows; thereafter $f_n'(y)$ automatically exists.

It is well known and indeed obvious that the general solution of

$$f_n'(x) = nf_{n-1}(x) \tag{1}$$

is the polynomial

$$f_n(x) = (x+\alpha)^n \equiv \sum_{s=0}^{n} \binom{n}{s} \alpha_{n-s} x^s,$$

where, as indicated, $(x+\alpha)^n$ is the *symbolic nth power* of $x+\alpha$, and α is the *umbra* of the sequence $\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots$ of arbitrary constants. That is, the general solution of (1) is the *Appell polynomial* of *rank n* with α as *base*. Note that the coefficient of x^n in $f_n(x)$ is α_0. We may refer to α as the *base* of the sequence $f_0(x), f_1(x), \cdots, f_n(x), \cdots$.

Let h be independent of both n and x, and let $\gamma(n)$ be a function of n alone such that $\gamma(0)\neq 0, \infty$; the excluded values lead only to trivialities.

Then if $g_n(x)$ is any solution of (1), the necessary and sufficient condition that $\gamma(n)\, g_n(x)$ shall be a solution of

$$\psi_n'(x) = \psi_{n-1}(x), \tag{2}$$

is $n\gamma(n) = \gamma(n-1)$; hence $\gamma(n) = \gamma(0)/n!$, and we have the following:

If k is an arbitrary constant which takes neither of the trivial values 0, ∞, and if $f_n(x) \equiv g_n(x)$ is the general solution of (1), *the general solution of* (2) *is*

$$\psi_n(x) \equiv kg_n(x)/n!\,; \tag{3}$$

conversely, the general solution of (1) *is* $f_n(x) \equiv n!\,\psi_n(x)/k$.

We shall call $\psi_n(x)$ the *canonical* polynomial of *degree* n with *base* α, and, when necessary, say that $\psi_n(x)$ *corresponds to* the Appell polynomial $g_n(x)$ from which it is constructed, and refer to α as the *base of the sequence* $\psi_0(x), \psi_1(x), \cdots, \psi_n(x), \cdots$.

Let $\tau(n)$, called the *multiplier*, be a function of n alone which is finite for all integers $n \geqq 0$, and not identically zero. Then, if there exist constants a, b, other than the trivial pair $(a, b) = (1, 0)$, such that

$$f_n(ax + b) = \tau(n) f_n(x), \tag{4}$$

we shall call

$$f_0(x), f_1(x), \cdots, f_n(x), \cdots$$

an *invariant sequence with respect to the transformation* $[x, ax+b]$, or briefly, an *invariant sequence*. When there can be no confusion between the term $f_n(x)$ and the sequence of which this is the nth term, we shall refer to the sequence as $f_n(x)$. Note particularly that in this definition the terms are not restricted to be polynomials.

If $f_n(x)$ for a particular $(a, b, \tau(n))$ is the general solution of (4), then $kf_n(x)/n!$, where k is an arbitrary constant, is also a solution. Hence the simultaneous solutions, if any, of (2), (4) are canonical polynomials. Let $f_n(x)$ for a particular $(a, b, \tau(n))$ be the general solution of

$$f_n'(x) = f_{n-1}(x), \qquad f_n(ax + b) = \tau(n) f_n(x). \tag{5}$$

Then we define

$$f_0(x), f_1(x), \cdots, f_n(x), \cdots$$

to be an *invariant sequence of polynomials* with the *characteristic* $(a, b, \tau(n))$, or simply an *invariant polynomial sequence*, when the characteristic is understood or otherwise indicated.

Our problem is to determine all invariant polynomial sequences. This will be accomplished when we find the multiplier $\tau(n)$, the constants a, b in

the transformation $[x, ax+b]$, and sufficiently define the bases of the Appell polynomials corresponding to the canonical polynomials concerned for a given characteristic. We shall first partially dispose of the multiplier. This depends upon the index, next defined, which is the taproot of the whole theory. When b is an integer, the sequences are of particular interest (§6, Theorem 10).

Let ϕ be the umbra of the sequence of absolute constants $\phi_0, \phi_1, \cdots, \phi_n, \cdots$. The least integer $s \geqq 0$ such that $\phi_s \neq 0$, will be called the *index* of ϕ.

If ϕ is the base of the sequence $f_n(x)$, and s is the index of ϕ, we replace n by $n+s$ in (5), differentiate the result n times successively, and get

$$a^n f_s(ax + b) = \tau(n + s) f_s(x).$$

Since $f_s(x) \equiv (x+\phi)^s = \phi_s = f_s(ax+b)$, we have $\tau(n+s) = a^n$. Again, since $f_j(x) = 0$ $(j = 0, 1, \cdots, s-1)$, it is immaterial what finite values be assigned to $\tau(j)$ $(j = 0, 1, \cdots, s-1)$. In particular we may take $\tau(j) = a^{j-s}(j = 0, 1, \cdots, s-1)$. Hence

$$\tau(n) \equiv a^{n-s} \tag{6}$$

is the value of the multiplier for the invariant polynomial sequence defined in (5), *in which* $f_n(x)$ *has base* ϕ, *and* s *is the index of* ϕ.

The classic instances of (5) are given by the polynomials whose bases are B, G, E, L, these being the umbrae of the sequences of the numbers of Bernoulli, Genocchi, Euler and Lucas. The customary manner of proving that these polynomials are indeed instances is somewhat fortuitous and effectively conceals the root of the matter, which is the index of the numerical sequence concerned in each case. Incidentally our general theorems give much more than the classic results for these instances. It will be interesting to observe the fundamental part played by the index in the general theory and in its applications to the classic instances.

2. **Equivalent sequences.** This section refers to the functional equation (4) in §1, so that $f_n(x)$ is not restricted to be a polynomial, and (6) does not necessarily hold.

The invariant sequences $f_n(x)$, $g_n(x)$ are defined to be *identical* if and only if $f_n(x) = g_n(x)$; otherwise they are *distinct*. Distinct sequences invariant with respect to the same given transformation $[x, ax+b]$ will be called *equivalent*. We now assign necessary and sufficient conditions for invariant sequences to be equivalent, and give the requisite formulas.

Let $a \neq 0$, $\alpha \neq 0$, b, β be constants other than $(a, b) = (\alpha, \beta) = (1, 0)$; the excluded values give only trivialities. Let r, s be constant integers $\geqq 0$, and

$f_n(x)$, $h_n(x)$ functions of x defined as in §1, beginning; also let $\tau(n)$ be as in (4), and $\sigma(n)$ a similarly defined function such that

$$f_n(ax + b) = \tau(n)f_n(x), \quad h_n(ax + b) = \sigma(n)h_n(x).$$

Let $c \neq 0$, $\gamma \neq 0$, d, δ be constants. If $\tau(n+r) \neq 0$, $\sigma(n+s) \neq 0$, we define $g_n(x)$, $k_n(x)$ by

$$g_n(x) \equiv f_{n+r}(cx + d), \qquad k_n(x) \equiv h_{n+s}(\gamma x + \delta).$$

Then, as may be easily verified, the functional equations (4) for g, k are

$$g_n\left(\frac{x}{a} + \frac{d - b - ad}{ac}\right) = \frac{1}{\tau(n + r)}g_n(x),$$

$$k_n\left(\frac{x}{\alpha} + \frac{\delta - \beta - \alpha\delta}{\alpha\gamma}\right) = \frac{1}{\sigma(n + s)}k_n(x).$$

We require the conditions upon the functions τ, σ and the several constants which shall yield two or more of the sequences

$$f_n(x), \quad g_n(x), \quad h_n(x), \quad k_n(x)$$

as solutions of a single functional equation of the type

$$\xi_n(\lambda x + \mu) = \rho(n + j)\xi_n(x),$$

in which j is a constant integer $\geqq 0$, $\rho(n+j)$ is defined, finite and not identically zero for all integers $n \geqq 0$; λ, μ are constants other than $(\lambda, \mu) = (1, 0)$, and $\xi_n(x)$ is single-valued and finite for all x considered.

Comparing the above functional equations for f, g, h, k, we find that there are precisely two distinct non-trivial solutions. Writing $(\alpha, \beta, \delta) \equiv (a, p, t)$ in the solutions thus found, we get the following:

THEOREM 1. *If* $ap \neq 0$, *and* $\tau(n+r) \neq 0$, *the equation*

$$\xi_n\left(\frac{x}{a} + p\right) = \frac{1}{\tau(n + r)}\xi_n(x)$$

has the solution

$$\xi_n(x) = g_n(x) \equiv f_{n+r}\left(\frac{[d(1 - a) - b]x}{ap} + d\right),$$

where d *is an arbitrary constant,* r *an arbitrary constant integer* $\geqq 0$, *and* $f_n(x)$ *is any solution of*

$$f_n(ax + b) = \tau(n)f_n(x).$$

THEOREM 2. *If* $ac \neq 0$ *and* $d(1-a) \neq b$, *and if and only if* $\tau(n+r) = \sigma(n+s) \neq 0$, *where* r, s *are integers* $\geqq 0$, *the equation*

$$\xi_n\left(\frac{x}{a} + \frac{d(1-a)-b}{ac}\right) = \frac{1}{\tau(n+r)}\xi_n(x)$$

has the solutions

$$\xi_n(x) = g_n(x) \equiv f_{n+r}(cx+d),$$

$$\xi_n(x) = k_n(x) \equiv h_{n+s}\left(\frac{c[t(a-1)+p]x}{d(a-1)+b} + t\right),$$

where d, t *are arbitrary constants and* $f_n(x)$, $h_n(x)$ *are any solutions of* $f_n(ax+b) = \tau(n)f_n(x)$, $h_n(ax+p) = \sigma(n)h_n(x)$.

Thus if in Theorem 1 a solution of each of the equations for ξ, f be known, g is a second solution of the ξ equation; if in Theorem 2 a solution for each of the f, h equations be known, the ξ equation has the two solutions g, k. The solutions can be easily verified. In §5 we find the equivalents of these theorems for invariant polynomial sequences and show how they are to be applied.

3. **Generators.** If $\phi_0, \phi_1, \cdots, \phi_n, \cdots$ is a sequence of numbers, real or complex, and z is a parameter, we shall call

$$e^{\phi z} \equiv \sum_0^\infty \phi_n(2^n/n!)$$

the *generator* of the sequence whose umbra is ϕ. The generator of the sequence of Appell polynomials in x with base ϕ is $e^{xz}e^{\phi z}$, or $e^{(x+\phi)z}$. Generators $e^{\phi z}$, $e^{\psi z}$ are defined to be *equal*, $e^{\phi z} = e^{\psi z}$, when and only when $\phi_n = \psi_n$. The symbolic or umbral calculus of such generators being well known, we may dispense with further details, except to remark that this calculus has been founded postulationally on an algebraic basis which renders all discussion of convergence in the use of generators for deriving relations between elements of sequences irrelevant.

Let $f_n(x)$ be the sequence of Appell polynomials with ϕ as base. Then

$$e^{(x+\phi)z} = e^{f(x)z}.$$

Let $g_n(x)$ be the sequence of canonical polynomials corresponding to $f_n(x)$. If there exist constants a, b, c, k, other than the trivial sets $(a, b, c, k) = (1, 0, 1, 1), (0, 0, 0, 0)$, such that

$$g_n(ax+b) = kc^n g_n(x),$$

we shall call $e^{\phi z}$ an *invariant generator*. If s is the index of ϕ, by §1(5), (6) we have

$$kc^n = \tau(n) = a^{n-s};$$

but it is more convenient in §4 to use kc^n instead of either of its equivalents $\tau(n)$, a^{n-s}. From the definitions in §1 it follows at once that the problem of determining all invariant polynomial sequences is identical with that of finding all invariant generators.

4. **Invariant generators.** Let u, v be independent variables, and $T(u, v)$ a function of u, v such that $T(z, e^z)$ is a generator as defined in §3. Let the index of ϕ be s; define ψ by

$$(n + s)!\psi_n \equiv n!\phi_{n+s},$$

and let the generator of ψ be $T(z, e^z)$. Then the index of ψ is zero, and

$$T(z,e^z) = e^{\psi z}, \qquad z^s T(z,e^z) = e^{\phi z}.$$

If now there exist constants (a, b, c, k) different from $(1, 0, 1, 1)$, $(0, 0, 0, 0)$ such that

$$e^{(ax+b)z}T(z,e^z) = kc^s e^{cxz}T(cz,e^z),$$

then and only then is $e^{\phi z}$ an invariant generator, as is evident on comparing the generators of $f_n(ax+b)$, $kc^n f_n(x)$, the notation being as in §3. Multiply throughout by e^{-cxz}. Then the new left member must be independent of x, since the new right is. Hence $a=c$, and we have

$$e^{bz}T(z,e^z) = ka^s T(az,e^{az}),$$

as the necessary and sufficient condition upon $T(u, v)$ in order that $T(z, e^z)$ shall be an invariant generator. From the last,

$$(\psi + b)^n = ka^{n+s}\psi^n.$$

Conversely, this implies the preceding equality, and hence it also is necessary and sufficient.

Reject the trivial cases $ka=0$. Take $n=0$, 1, 2 in the last. Then we get

$$ka^s = 1, \qquad b = (a-1)\psi_1/\psi_0, \qquad (a^2-1)(\psi_0\psi_2 - \psi_1^2) = 0.$$

Excluding $a=1$, which yields merely the identical transformation $[x, x]$, and noting, as is easily seen, that $\psi_0\psi_2 - \psi_1^2 = 0$ leads only to trivialities, we get the unique solution

$$(a,b,k) = (-1, -2\phi_{s+1}/[(s+1)\phi_s], (-1)^s).$$

THEOREM 3. *There exists precisely one non-trivial characteristic* $(a, b, \tau(n))$ *such that the sequence of canonical polynomials* $f_n(x)$ *with the base* ϕ *has the property*

$$f_n(ax + b) = \tau(n) f_n(x) ;$$

if s *is the index of* ϕ *the characteristic is*

$$(a,b,\tau(n)) = (-1, -2\phi_{s+1}/[(s+1)\phi_s], (-1)^{n+s}),$$

and ϕ *is generated by* $R(z,e^z)$, *where* $R(u,v)$ *is any solution of the functional equation*

$$v^b R(u,v) = (-1)^s R(-u, v^{-1}),$$

in which u, v *are independent variables and* b *is as above.*

From this we have

COROLLARY 1. *When the index* s *of* ϕ *is given, the first* $s+2$ *terms of* ϕ *are necessary and sufficient to determine the transformation with respect to which the invariant polynomial sequence with base* ϕ *is invariant, and its characteristic.*

COROLLARY 2. *A particular invariant polynomial sequence and its characteristic are uniquely determined by the generator of the base of the polynomials; conversely, a particular generator determines precisely one invariant polynomial sequence and its unique characteristic.*

The distinction between the information furnished by these two corollaries may be emphasized: the functional equations §1 (5), for a given characteristic $(a, b, \tau(n))$, are determined by the first $s+2$ terms only of the base, and have an infinity of solutions; to select from this infinity a particular solution it is necessary to know the generator of the base, not merely the first $s+2$ terms of the sequence which it generates.

To obtain an *element* of the solution of the functional equation of invariant generators, we assume that $R(u, v)$ is a sum of terms of the form $v^{\lambda b+\mu} A(u)$, where λ, μ are constants and $A(u)$ is independent of v.

THEOREM 4. *The functional equation of invariant generators*

$$v^b R(u,v) = (-1)^s R(-u, v^{-1}),$$

in which u, v *are independent variables,* b *is an arbitrary constant and* s *is an integer* $\geqq 0$, *has the elementary solution, involving both* u *and* v, *or only* v,

$$R(u,v) \equiv v^{rb+t} F(u) + v^{-(r+1)b-t}(-1)^s F(-u),$$

in which r, t *are arbitrary constants and* $F(u)$ *is an arbitrary function of* u *alone, including the case* $F(u)$ *constant.*

By requiring given rational functions of $v, v^{rb}, v^{tb}, \cdots$ and solutions of the equation to be further solutions, we easily obtain the following:

THEOREM 5. *All rational functions of given solutions of the functional equation of invariant generators that contain both variables u, v and are again solutions, can be constructed by repetitions of the operations indicated in*

$$kR(u,v), \quad R_1(u,v) + R_2(u,v),$$
$$(1 + (-1)^s)v^{\epsilon b/2}R_1(u,v)[R_2(u,v)]^\epsilon \qquad (\epsilon = 1, -1),$$

where b, s are as in Theorem 4, *k is an arbitrary constant, and* $R(u, v)$, $R_1(u, v)$, $R_2(u, v)$ *are given solutions.*

In applying this and the next, the usual precautions regarding vanishing functions as divisors are to be observed. The separable solutions are of some interest, as the four classic instances of invariant sequences mentioned in §1 have generators of this type.

THEOREM 6. *All solutions of the functional equation of invariant generators of the type* $A(u)\,B(v)$, *where* $A(u), B(v)$ *are functions of u alone, v alone, are given by*

$$A(u) = F(u) + (-1)^\eta F(-u), \qquad B(v) = G(v) \qquad (\eta = 0,1),$$

where $F(u)$ *is an arbitrary function of u, and* $G(v)$ *is any solution of*

$$G(v^{-1}) = (-1)^{(\eta+1)s}v^bG(v),$$

where the same value of η *is to be used in both of* $A(u)$, $B(v)$. *The G equation has the elementary solution*

$$G(v) \equiv v^{rb+t} + (-1)^{(\eta+1)s}v^{-(r+1)b-t},$$

where r, t are arbitrary constants; all rational functions of given solutions that are again solutions can be constructed according to repetitions of the operations indicated in

$$kG(v), \quad G_1(v) + G_2(v), \quad (-1)^{(\eta+1)s}v^{\epsilon b/2}G_1(v)[G_2(v)]^\epsilon \qquad (\epsilon = 1, -1),$$

where k is an arbitrary constant, and $G(v)$, $G_1(v)$, $G_2(v)$ *are given solutions.*

COROLLARY 3. *Solutions of the type* $A(u)$ *exist only when* $b = 0$; $A(u)$ *is then as in Theorem* 6 *with* $\eta = 1$. *All solutions of the type* $B(v)$ *are obtained from Theorem* 6 *by taking* $\eta = 0$ *in* $B(v)$ *as there.*

The first part of this has some interesting consequences. Since $F(z) +$

$(-1)^s F(-z)$ is here the generator of an invariant polynomial sequence whose base ϕ is of index s, we have the formal expansion

$$F(z) + (-1)^s F(-z) = z^s \sum_{n=0}^{\infty} \phi_{2n+s} \frac{z^{2n}}{(2n+s)!},$$

since the left is changed into $(-1)^s$ times itself when z is replaced by $-z$, and $\phi_{2n+s+1}=0$. The characteristic is here $(-1, 0, (-1)^{n+s})$. If s is even, this becomes $(-1, 0, (-1)^n)$ which, as will be seen in §8, is the characteristic of each of the invariant polynomial sequences whose respective bases are E, L (cf. §1). Thus extensive tracts of the theories of the ϕ, E, L invariant polynomial sequences will be identical. One respect in which they may differ is more striking. By considering the special case in which the generator of ϕ is a rational function of z of the most general type possible, we easily find the following:

COROLLARY 4. *Let α, β be arbitrary constant integers, $(\alpha, \beta) \neq (0, 0)$, and the p_i, q_i $(i=0, 1, \cdots, \alpha;\ j=0, 1, \cdots, \beta)$ arbitrary constants; $(p_0, q_0) \neq (0, 0)$. Then ϕ defined by*

$$\phi_{2n+s+1} = 0, \qquad \sum_{j=0}^{\beta} q_j \frac{\phi_{2n+s-j}}{(2n+s-j)!} = \theta_n p_n,$$

where $\theta_n = 1$ or 0 according as $n \leqq \alpha$ or $n > \alpha$, is the base of an invariant polynomial sequence with index $(-1, 0, (-1)^{n+s})$.

Neither of E, L can be defined by a linear difference equation of constant order, since otherwise certain general circular functions would be algebraic.

Continuing with the general theorems we exhaust the possibilities in the next.

THEOREM 7. *All solutions of the functional equation of invariant generators of the types $A(u)\ R(u, v)$, $B(v)\ R(u, v)$, where $R(u, v)$ is any solution involving both u and v, and $A(u)$, $B(v)$ are functions of u alone, v alone respectively, are given by $A(u) = F(u) + F(-u)$, where $F(u)$ is an arbitrary function of u, and by $B(v) = H(v)$, where $H(v)$ is any solution of $H(v^{-1}) = H(v)$; the H equation has the elementary solution $H(v) \equiv k(v^r + v^{-r})$, where k, r are arbitrary constants, and $kH(u)$, $H_1(u) + H_2(u)$, $H_1(u)\ H_2(u)$, $H_1(u)/H_2(u)$, where $H(u)$, $H_1(u)$, $H_2(u)$ are given solutions, are further solutions.*

5. **Equivalent invariant polynomial sequences.** Applying §4 Theorem 3 to §2 Theorems 1, 2 we get the complete solution of the problem of equivalence for invariant polynomial sequences.

THEOREM 8. *Let the Appell polynomials $f_n(x)$, $h_n(x)$ have the respective bases λ, μ whose indices are s,t, and let the corresponding canonical polynomials be $F_n(x)$, $H_n(x)$, so that*

$$n!F_n(x) \equiv pf_n(x), \qquad n!\,H_n(x) \equiv qh_n(x),$$

where $p \neq 0, q \neq 0$ are arbitrary constants. Then

$$F_n'(x) = F_{n-1}(x), \qquad F_n(-x + b) = (-1)^{n+s}F_n(x),$$
$$H_n'(x) = H_{n-1}(x), \qquad H_n(-x + c) = (-1)^{n+t}H_n(x),$$
$$b \equiv -2\lambda_{s+1}/[(s+1)\lambda_s], \qquad c \equiv -2\mu_{t+1}/[(t+1)\mu_t].$$

Let i, j, m be arbitrary constant integers $\geqq 0$. Then

$$X_n(x) = G_n(x) \equiv F_{n+2i+s+m}\left(x + \frac{b-a}{2}\right),$$

$$X_n(x) = K_n(x) \equiv H_{n+2j+t+m}\left(x + \frac{c-a}{2}\right)$$

are solutions of

$$X_n'(x) = X_{n-1}(x), \quad X_n(-x + a) = (-1)^{n+m}X_n(x),$$

where a is an arbitrary constant, being equal to $-2\sigma_{m+1}/[(m+1)\sigma_m]$, where σ is the base of the general Appell polynomial in x and m is the index of σ.

Thus, according to the definition in §2, the G, K sequences are equivalent. Let us call the X sequence the *equalizing sequence* for G, K. The data in a specific application of Theorem 8 will be the f, h sequences. Without the theorem the invariant properties of these sequences must be investigated separately. The advantages of replacing f, h by F, H which are equivalent are obvious. By successive applications we get the following general result:

THEOREM 9. *By repeated applications of Theorem 8 to the equalizing sequences of pairs of sequences of Appell polynomials, and to an equalizing sequence and a sequence of canonical polynomials corresponding to a given sequence of Appell polynomials, any number of sequences of Appell polynomials can be transformed into the same number of equivalent sequences, all equalized with respect to one invariant polynomial sequence.*

An example is given in §8.

COROLLARY 5. *The equalizing sequence in Theorem 8 is unique, as also are the equivalent sequences which it equalizes, up to an arbitrary constant a in the argument of the equalizing sequence, and arbitrary constant integers $\geqq 0$ in the ranks of the polynomials equalized, and the arbitrary constant a in their arguments.*

6. **Rational invariant polynomial sequences.** If the generator $R(z, e^z)$ of ϕ is such that $R(u, v)$ is a rational function of the independent variables u, v, we shall call the invariant polynomial sequence with ϕ as base *rational*. The base of a rational invariant polynomial sequence will be called *rational*. The terms of a rational base are of course not necessarily rational numbers. The determination of all rational invariant polynomial sequences is reduced by the next theorem to that of all rational bases, which is done in §7. From §4 Theorem 3 we get the following:

THEOREM 10. *The set of all rational invariant polynomial sequences is completely and uniquely defined by the properties*

$$f'(x) = f_{n-1}(x), \quad f_n(-x + 2\phi_{s+1}/[(s+1)\phi_s]) = (-1)^{n+s} f_n(x),$$

where ϕ, whose index is s, is generated by $R(u, v)$, where $R(u, v)$ is any rational function of the independent variables u, v which is such that

$$v^b R(u,v) = (-1)^s R(-u, v^{-1}), \qquad b \equiv -2\phi_{s+1}/[(s+1)\phi_s],$$

and hence, for all such sequences, $-2\phi_{s+1}/[(s+1)\phi_s]$ is an integer.

7. **Rational bases.** Write the $R(u, v)$ of Theorem 10 in the form $R(u, v) \equiv v^c N(u, v)/D(u, v)$, where

$$N(u,v) \equiv \sum_{i=0}^{\alpha} v^i N(u), \qquad D(u,v) \equiv \sum_{j=0}^{\beta} v^j D(u)$$

are polynomials in u, v with highest common factor unity, the $N_i(u)$, $D_j(u)$ are polynomials in u alone of degree $\gtreqless 0$, and c is a constant integer $\gtreqless 0$. The trivial case in which $R(u, v)$ is a constant is excluded. The case $\alpha = \beta = c = 0$ was discussed in §4, Theorem 6, Corollary 4; it is included in the next theorem. As the solution of the pertinent functional equation presents no difficulty, we merely state the result, which can be verified by inspection.

THEOREM 11. *The general solution of the functional equation of rational generators,*

$$v^b R(u,v) = (-1)^s R(-u, v^{-1}),$$

in which b, s are arbitrary constant integers $\gtreqless 0$, and $R(u, v)$ is a rational function of the independent variables u, v is

$$R(u,v) = v^c N(u,v)/D(u,v),$$

where

c is an arbitrary constant intger $\gtreqless 0$;

$$N(u,v) = \sum_{i=0}^{\alpha}[v^iN_i(u) + \epsilon(-1)^{\eta s}v^{\alpha-i}N_i(-u)],$$

$$D(u,v) = \sum_{j=0}^{\beta}[v^jD_j(u) + \epsilon(-1)^{(\eta+1)s}v^{\beta-j}D_j(-u)];$$

α, β are arbitrary constant integers such that

$$\alpha\beta \neq 0, \quad \alpha \geqq 0, \quad \beta \geqq 0, \quad \beta - \alpha = b + 2c;$$

no one of $N_0(u)$, $N_\alpha(u)$, $D_0(u)$, $D_\beta(u)$ is identically zero;

$N_i(u)$ $(i=0, \cdots, \alpha)$, $D_j(u)$ $(j=0, \cdots, \beta)$, with the exceptions just noted, are arbitrary polynomials in u alone of any degrees $\geqq 0$;

ϵ is a definite one of $1, -1$, and η a definite one of $0, 1$, and the same value of (ϵ, η) is to be used in both of $N(u,v), D(u,v)$.

According to the values of (ϵ, η), rational invariant polynomial sequences fall into four mutually exclusive sets,each of which contains an infinity of sequences. The like holds when the numbers of the base are restricted to be rational, or to be in any given number field.

8. **The classic instances.** In the even suffix notation the sequences whose umbrae are B, G, E, L (cf. §1, end), have all terms, except B_1, G_1, of odd ranks, zero, and the signs alternate after the first term (rank 0).

The initial values necessary and sufficient for our purpose are

$$(B_0,B_1) = (1, -\tfrac{1}{2}), \qquad (G_0,G_1,G_2) = (0,1,-1),$$
$$(E_0,E_1) = (1,0), \qquad (L_0,L_1) = (\tfrac{1}{2},0);$$

hence the indices s of B, G, E, L are $0, 1, 0, 0$ respectively, and the corresponding Appell polynomials are

$$\beta_n(x) \equiv (x+B)^n, \qquad \gamma_n(x) \equiv (x+G)^n,$$
$$\eta_n(x) \equiv (x+E)^n, \qquad \lambda_n(x) \equiv (x+L)^n.$$

From the stated values of s and the first $s+2$ initial values in each case we find the values of b (§4, Theorem 3) for $\beta, \gamma, \eta, \lambda$ to be $1, 1, 0, 0$ respectively. Hence, by Theorem 3, we have

THEOREM 12. *The $\beta, \gamma, \eta, \lambda$ sequences of polynomials are solutions of the respective pairs of functional equations*

$$\beta_n'(x) = n\beta_{n-1}(x), \qquad \beta_n(-x+1) = (-1)^n\beta_n(x);$$
$$\gamma_n'(x) = n\gamma_{n-1}(x), \qquad \gamma_n(-x+1) = (-1)^{n+1}\gamma_n(x);$$
$$\eta_n'(x) = n\eta_{n-1}(x), \qquad \eta_n(-x) = (-1)^n\eta_n(x);$$
$$\lambda_n'(x) = n\lambda_{n-1}(x), \qquad \lambda_n(-x) = (-1)^n\lambda_n(x).$$

The known generators of B, G, E, L are (in the u, v form)

$$\frac{u}{v-1}, \quad \frac{2u}{v+1}, \quad \frac{2v}{v^2+1}, \quad \frac{uv}{v^2-1}$$

respectively, and these obviously verify §7, Theorem 11. [In particular, the integers α, β, b, c in Theorem 11 are here (there can be no confusion between these α, β and the umbrae)

$$(\alpha,\beta,b,c) = (0,1,1,0), \quad (0,1,1,0), \quad (0,2,0,1), \quad (0,2,0,1),$$

respectively ($c \equiv$ the exponent $\gtrless 0$ of the highest or the lowest power of v dividing the numerator; $\alpha \equiv$ the degree of the numerator in v after the division; $\beta \equiv$ the degree in v of the denominator; $b \equiv$ the integer already used in writing down the functional equations); the sign of c is so chosen that the denominator is not divisible by v].

To equalize the β, γ pair by §5 Theorem 8, take $\beta \equiv f$, $\gamma \equiv h$, and hence $(s, t) = (0, 1)$, $c = b = 1$, in the notation of §5. This gives

THEOREM 13. *If i, j, m are arbitrary constant integers $\geqq 0$, and p, q, a arbitrary constants, and if*

$$(n+2i+m)!\,G_n(x) \equiv p\beta_{n+2i+m}\left(x + \frac{1-a}{2}\right),$$

$$(n+2j+m+1)!\,K_n(x) \equiv q\gamma_{n+2j+m+1}\left(x + \frac{1-a}{2}\right),$$

then $X_n(x) = G_n(x)$, $X_n(x) = K_n(x)$ are solutions of

$$X_n'(x) = X_{n-1}(x), \quad X_n(-x+a) = (-1)^{n+m}X_n(x).$$

In the same way we find for the η, λ pair

THEOREM 14. *If k, l, t are arbitrary constant integers $\geqq 0$, and r, g, b arbitrary constants, and if*

$$(n+2k+t)!\,P_n(x) \equiv r\eta_{n+2k+t}\left(x - \frac{b}{2}\right),$$

$$(n+2l+t)!\,Q_n(x) \equiv g\lambda_{n+2l+t}\left(x - \frac{b}{2}\right),$$

then $Y_n(x) = P_n(x)$, $Y_n(x) = Q_n(x)$ are solutions of

$$Y_n'(x) = Y_{n-1}(x), \quad Y_n(-x+b) = (-1)^{n+t}Y_n(x).$$

We now apply §5, Theorem 9. First equalize X, Y.

THEOREM 15. *If d, e, s are arbitrary constant integers $\geqq 0$, and w, h, c arbitrary constants, and if*

$$(n+2d+m+s)!S_n(x) \equiv wX_{n+2d+m+s}\left(x+\frac{a-c}{2}\right),$$

$$(n+2e+t+s)!T_n(x) \equiv hY_{n+2e+t+s}\left(x+\frac{b-c}{2}\right),$$

then $Z_n(x)=S_n(x)$, $Z_n(x)=T_n(x)$ are solutions of

$$Z_n'(x) = Z_{n-1}(x), \qquad Z_n(-x+c) = (-1)^{n+s}Z_n(x).$$

Replace $S_n(x)$ by its equivalent in terms of β, γ as given by combining the definitions in Theorems 13, 15, and similarly for $T_n(x)$, η, λ and Theorems 14, 15. Then finally we have the general equalization of β, γ, η, λ.

THEOREM 16. *If i, j, k, l, s are arbitrary constant integers $\geqq 0$, and p, q, r, g, c are arbitrary constants, and if*

$$(n+2i+s)!B_n(x) \equiv p\beta_{n+2i+s}\left(x+\frac{1-c}{2}\right),$$

$$(n+2j+s+1)!G_n(x) \equiv q\gamma_{n+2j+s+1}\left(x+\frac{1-c}{2}\right),$$

$$(n+2k+s)!E_n(x) \equiv r\eta_{n+2k+s}\left(x-\frac{c}{2}\right),$$

$$(n+2l+s)!L_n(x) \equiv g\lambda_{n+2l+s}\left(x-\frac{c}{2}\right),$$

then $W_n(x)=B_n(x)$, $W_n(x)=G_n(x)$, $W_n(x)=E_n(x)$, $W_n(x)=L_n(x)$ are solutions of

$$W_n'(x) = W_{n-1}(x), \qquad W_n(-x+c) = (-1)^{n+s}W_n(x).$$

To exhibit the particular form of this current for $B_n(x)$, $G_n(x)$ in the literature, we state the following

COROLLARY 6. *The equations*

$$U_n'(x) = U_{n-1}(x), \qquad U_n(-x-1) = (-1)^n U_n(x)$$

have the solutions

$$U_n(x) = \mathfrak{B}_n(x) \equiv \beta_n(x+1)/n!, \qquad U_n(x) = \mathfrak{E}_n(x) \equiv \eta_n(x+\tfrac{1}{2})/n!,$$

$$U_n(x) = \mathfrak{G}_n(x) \equiv \gamma_{n+1}(x+1)/(n+1)!, \qquad U_n(x) = \mathfrak{L}_n(x) \equiv \lambda_n(x+\tfrac{1}{2})/n!.$$

Glancing back over this section we see that everything in it, with the exception of the u, v forms of the generators, is an immediate consequence of the numerical values of the indices s and the first $s+2$ terms of the bases of the polynomials. That this should be so, and that the like holds also in the general case, is a remarkable simplification of the theory.

For those who may wish to pursue the β, γ, η, λ, and hence also the B, G, E, L, further by the methods of this paper, we add

$$2(1-2^n)B_n = G_n, \qquad (1-2^{n-1})B_n = L_n,$$

$$(2G)^n = 2n(E-1)^{n-1}, \qquad 2nE^{n-1} = (2G+1)^n,$$

all of which are well known and follow at once from trivial algebraic identities between the generators in their u, v forms.

9. **Remarks on notation and method.** Many writers on the Bernoulli and Euler numbers prefer a notation which makes the use of the symbolic method impossible, for example Nielsen in his *Traité Elémentaire des Nombres de Bernoulli* (Paris, 1923, pp. 9+398). His $(-1)^{n-1}B_n$, $n(-1)^n 2^{2(1-n)}T_n$ are our B_{2n}, G_{2n} $(n>0)$; his polynomials $2(n+1)!\,E_n(x)$, $n!B_n(x)$ are our $(x+G+1)^{n+1}$, $(x+B+1)^n$. By ignoring the well established symbolic method he is compelled (loc. cit., p. 46) to write his $E_n(x)$ in the form

$$\frac{1}{2}\frac{x^n}{n!} + \sum_{s=1}^{s\leq(n+1)/2} \frac{(-1)^{s-1}T_s x^{n-2s+1}}{(2s-1)!(n-2s+1)!2^{2s}},$$

which seems less suggestive and less tractable than its equivalent

$$(x+G+1)^{n+1}/[2(n+1)!].$$

As Neilsen in his preface emphasizes that the use of the functional equations is a "méthode élémentaire qui est beaucoup plus fondamentale que la méthode symbolique, développée notamment par Lucas" (*it was invented and very extensively applied to the Bernoulli and Euler numbers by J. Blissard fifteen years before Lucas' work was published*), it is well to point out what is indeed otherwise self-evident: *neither method is more fundamental than the other in any significant sense; they are abstractly identical.* For, the symbolic method, as we have shown, leads directly to the functional equations, and these are uniquely determined by the numerical values of s and the first $s+2$ terms of the respective bases, but not without them; conversely, the functional equations, together with the numerical values of s and the first $s+2$ terms of the bases, uniquely determine the generators, which are the fundamental formulas of the symbolic method in any given particular instances. That is, *each method implies and is implied by the other; they are*

thus formally equivalent in the sense of mathematical logic, or abstractly identical, as the term is used in algebra. To select a particular sequence given by the functional equations, the "elementary method" (to use Neilsen's name for it) adjoins a difference equation; the symbolic method presents the generator of the base, and again these procedures are abstractly identical.

Further, *the symbolic method, including the generators, is no more transcendental,* as has been carelessly alleged by certain writers, *than is the elementary.* For, the equality of generators is precisely matric equality, and this is exactly as transcendental as is mathematical induction, without which no formula inferred from the processes of the elementary method is proved, however obvious it may appear that the tedious induction will sustain the inference. Operations on generators are equivalent to the Cauchy addition, subtraction, multiplication and division of one-rowed matrices or, if preferred, of sequences, and these operations are abstractly identical with those of the elementary method. Heuristically, however, the advantage is with the symbolic method. This is abundantly evident on historical grounds, and is not affected by "elementary" reconstructions of theorems already known.

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